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On partial inverse operations in the class of preradicals of modules

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Abstract

In the present work two partial operations in the class of preradicals $\mathbb{PR}$ of the category R-Mod of left R-modules are defined and investigated. They are inverse operations for product with respect to meet and coproduct with respect to join. The criteria of existence of such operations are indicated. Main properties of this operation and relations with the lattice operations in $\mathbb{PR}$ are shown. Some particular cases are mentioned.

1 Introduction and preliminary facts

This work is devoted to the theory of radicals of modules ([1]-[5]) and contains the definitions and the investigations of two new operations in the class of preradicals of a module category.

Let R be a ring with unity and R-Mod be the category of unitary left R-modules. We remind that a *preradical* r of R-Mod is a subfunctor of identity functor of R-Mod, i.e. r associates to every module $M \in R$-Mod a submodule $r(M) \subseteq M$ such that $f(r(M)) \subseteq r(M')$ for every R-morphism $f : M \to M'$.

We denote by $\mathbb{PR}$ the class of all preradicals of the category R-Mod. In this class four operation are defined [1]:

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1) the *meet* $\underset{\alpha\in\mathfrak{A}}{\wedge} r_\alpha$ of a family of preradicals $\{r_\alpha\}_{\alpha\in\mathfrak{A}}$:

$$\left(\underset{\alpha\in\mathfrak{A}}{\wedge} r_\alpha\right)(M) \overset{def}{=} \bigcap_{\alpha\in\mathfrak{A}} r_\alpha(M),\ M\in R\text{-Mod};$$

2) the *join* $\underset{\alpha\in\mathfrak{A}}{\vee} r_\alpha$ of a family of preradicals $\{r_\alpha\}_{\alpha\in\mathfrak{A}}$:

$$\left(\underset{\alpha\in\mathfrak{A}}{\vee} r_\alpha\right)(M) \overset{def}{=} \sum_{\alpha\in\mathfrak{A}} r_\alpha(M),\ M\in R\text{-Mod};$$

3) the *product* $r\cdot s$ of preradicals $r,s\in\mathbb{PR}$:

$$(r\cdot s)(M) \overset{def}{=} r(s(M)),\ M\in R\text{-Mod}\ ;$$

4) the *coproduct* $r\,\#\,s$ of preradicals $r,s\in\mathbb{PR}$:

$$[(r\,\#\,s)(M)]/s(M) \overset{def}{=} r(M/s(M)\ ),\ M\in R\text{-Mod}.$$

In the class $\mathbb{PR}$ the partial order relation " $\leq$ " is defined by the rule:

$$r_1\leq r_2 \overset{def}{\Leftrightarrow} r_1(M)\subseteq r_2(M) \ \text{ for every }\ M\in R\text{-Mod}.$$

The class $\mathbb{PR}$ is a large complete lattice with respect to the operations of meet and join.

We remark that in the book [1] the coproduct is denoted by $(r:s)$ and is defined by the rule $[(r:s)(M)]/r(M) = s(M/r(M)\ )$, so $(r\,\#\,s)=(s:r)$.

The following properties of distributivity hold ([1]-[5]):

(1) $(\wedge\, r_\alpha)\cdot s=\wedge\,(r_\alpha\cdot s)$; (2) $(\vee\, r_\alpha)\cdot s=\vee\,(r_\alpha\cdot s)$;

(3) $(\wedge\, r_\alpha)\,\#\,s=\wedge\,(r_\alpha\,\#\,s)$; (4) $(\vee\, r_\alpha)\,\#\,s=\vee\,(r_\alpha\,\#\,s)$

for every family $\{r_\alpha\}_{\alpha\in\mathfrak{A}}\subseteq\mathbb{PR}$ and $s\in\mathbb{PR}$.

Using these relations some new inverse operations can be defined in the class $\mathbb{PR}$. Two of them, the left quotient with respect to join and the left coquotient with respect to meet, have been defined and investigated in [11] and [12]. In this work we will study other two inverse operations, namely, the left quotient with respect to meet and the left coquotient with respect to join. Similar questions are discussed in [8]-[10].

Now we remind the principal types of preradicals. A preradical $r\in\mathbb{PR}$ is called:

- *idempotent preradical*, if $r(r(M))=r(M)$ for every $M\in R$-Mod (or if $r\cdot r=r$);
- *radical*, if $r(M/r(M))=0$ for every $M\in R$-Mod (or if $r\,\#\,r=r$);
- *idempotent radical*, if both previous conditions are fulfilled;
- *prime*, if $r\neq 1$ and for any $t_1,t_2\in\mathbb{PR}$, $t_1\cdot t_2\leq r$ implies $t_1\leq r$ or $t_2\leq r$ [6];
- *coprime*, if $r\neq 0$ and for any $t_1,t_2\in\mathbb{PR}$, $t_1\,\#\,t_2\geq r$ implies $t_1\geq r$ or $t_2\geq r$ [7];

- $\wedge$-*prime*, if for any $t_1, t_2 \in \mathbb{PR}$, $t_1 \wedge t_2 \leq r$ implies $t_1 \leq r$ or $t_2 \leq r$ [6];
- $\vee$-*coprime*, if for any $t_1, t_2 \in \mathbb{PR}$, $t_1 \vee t_2 \geq r$ implies $t_1 \geq r$ or $t_2 \geq r$ [7];
- *irreducible*, if for any $t_1, t_2 \in \mathbb{PR}$, $t_1 \wedge t_2 = r$ implies $t_1 = r$ or $t_2 = r$ [6];
- *coirreducible*, if for any $t_1, t_2 \in \mathbb{PR}$, $t_1 \vee t_2 = r$ implies $t_1 = r$ or $t_2 = r$ [7].

The operations of meet and join are commutative and associative, while the operations of product and coproduct are associative. By means of these operations four preradicals are obtained which are arranged in the following order:

$$r \cdot s \leq r \wedge s \leq r \vee s \leq r \,\#\, s$$

for every $r, s \in \mathbb{PR}$.

During this work we will use the following facts and notions from general theory of preradicals (see [1]−[7]).

Lemma 1.1. *(Monotony of the product) For any $s_1, s_2 \in \mathbb{PR}$, $s_1 \leq s_2$ implies that $r \cdot s_1 \leq r \cdot s_2$ and $s_1 \cdot r \leq s_2 \cdot r$ for every $r \in \mathbb{PR}$.* □

Lemma 1.2. *(Monotony of the coproduct) For any $s_1, s_2 \in \mathbb{PR}$, $s_1 \leq s_2$ implies that $r \,\#\, s_1 \leq r \,\#\, s_2$ and $s_1 \,\#\, r \leq s_2 \,\#\, r$ for every $r \in \mathbb{PR}$.* □

Lemma 1.3. *For every $r, s, t \in \mathbb{PR}$ we have:*

1) $(r \cdot s) \,\#\, t \geq (r \,\#\, t) \cdot (s \,\#\, t)$;

2) $(r \,\#\, s) \cdot t \leq (r \cdot t) \,\#\, (s \cdot t)$. □

Definition 1.1. *The* equalizer *of preradical r is the preradical*

$$e(r) = \wedge \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot r = r \}.$$

Definition 1.2. *The* co-equalizer *of preradical r is the preradical*

$$c(r) = \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \,\#\, r = r \}.$$

2 Left quotient with respect to meet

Now we introduce and investigate the inverse operation of product with respect to meet in the class of preradicals $\mathbb{PR}$ of category R-Mod.

Definition 2.1. *Let $r, s \in \mathbb{PR}$. The* left quotient with respect to meet of r by s *is defined as the least preradical among $r_\alpha \in \mathbb{PR}$ with the property $r_\alpha \cdot s \geq r$. We denote this preradical by $r \,\overset{\wedge}{/}\, s$.*

We will call r the *numerator* and s the *denominator* of the quotient $r \,{}^{\wedge}\!\!/.\, s$.

Now we will study question on the existence of the left quotient with respect to meet.

Lemma 2.1. *Let $r, s \in \mathbb{PR}$. The left quotient $r \,{}^{\wedge}\!\!/.\, s$ of r by s with respect to meet exists if and only if $r \leq s$ and it can be presented in the form $r \,{}^{\wedge}\!\!/.\, s = \wedge \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s \geq r \}$.*

Proof. ($\Rightarrow$) If the left quotient $r \,{}^{\wedge}\!\!/.\, s$ exists, then there exists $r_\alpha \in \mathbb{PR}$ such that $r_\alpha \cdot s \geq r$. Since $1 \geq r_\alpha$, from the monotony of product we have $1 \cdot s \geq r_\alpha \cdot s$, therefore $s \geq r$.

($\Leftarrow$) Let $r \leq s$. Then $1 \cdot s = s \geq r$, therefore the family of preradicals $\{ r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s \geq r \}$ is not empty. So we can consider the preradical $\wedge \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s \geq r \}$, for which by distributivity of product relative to meet we obtain $\left(\bigwedge\limits_{r_\alpha \cdot s \geq r} r_\alpha \right) \cdot s = \bigwedge\limits_{r_\alpha \cdot s \geq r} (r_\alpha \cdot s) \geq r$. By the construction it is clear that the preradical $\bigwedge\limits_{r_\alpha \cdot s \geq r} r_\alpha$ is the least preradical of $\mathbb{PR}$ with property $r_\alpha \cdot s \geq r$. Therefore we have $r \,{}^{\wedge}\!\!/.\, s = \wedge \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s \geq r \}$. $\square$

Moreover, by the proof of Lemma 2.1 we have that $(r \,{}^{\wedge}\!\!/.\, s) \cdot s \geq r$, what we will often use further.

Lemma 2.2. *Let $r, s \in \mathbb{PR}$ and $r \leq s$. Then $r \,{}^{\wedge}\!\!/.\, s \geq r$.*

Proof. The condition $r \leq s$ ensures the existence of the left quotient $r \,{}^{\wedge}\!\!/.\, s$. Since $r \,{}^{\wedge}\!\!/.\, s \geq (r \,{}^{\wedge}\!\!/.\, s) \cdot s$, by the definition of the left quotient $(r \,{}^{\wedge}\!\!/.\, s) \cdot s \geq r$, it follows that $r \,{}^{\wedge}\!\!/.\, s \geq r$. $\square$

The next two statements show the concordance of the left quotient $r \,{}^{\wedge}\!\!/.\, s$ with the order relation $(\leq)$ of $\mathbb{PR}$.

Proposition 2.3. *(Monotony in the numerator) Let $r_1, r_2 \in \mathbb{PR}$ and $r_1 \leq r_2$. Then for every preradical $s \geq r_2$ we have $r_1 \,{}^{\wedge}\!\!/.\, s \leq r_2 \,{}^{\wedge}\!\!/.\, s$.*

Proof. By Lemma 2.1 there exist the left quotients $r_1 \,{}^{\wedge}\!\!/.\, s$, $r_2 \,{}^{\wedge}\!\!/.\, s$ and $r_1 \,{}^{\wedge}\!\!/.\, s = \wedge \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s \geq r_1 \}$, $r_2 \,{}^{\wedge}\!\!/.\, s = \wedge \left\{ r'_\beta \in \mathbb{PR} \,\middle|\, r'_\beta \cdot s \geq r_2 \right\}$.

Let $r_1 \leq r_2$ and $r'_\beta \cdot s \geq r_2$. Then $r'_\beta \cdot s \geq r_1$, so each r'_β is one of preradicals r_α. Therefore $\wedge \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s \geq r_1 \} \leq \wedge \left\{ r'_\beta \in \mathbb{PR} \,\middle|\, r'_\beta \cdot s \geq r_2 \right\}$, i.e. $r_1 \,{}^{\wedge}\!\!/.\, s \leq r_2 \,{}^{\wedge}\!\!/.\, s$. $\square$

Proposition 2.4. *(Antimonotony in the denominator) Let $s_1, s_2 \in \mathbb{PR}$ and $s_1 \leq s_2$. Then for every preradical $r \leq s_1$ we have $r \,{}^{\wedge}\!\!/.\, s_1 \geq r \,{}^{\wedge}\!\!/.\, s_2$.*

Proof. By Lemma 2.1 there exist the left quotients $r \;{}^{\wedge}\!/.\; s_1$, $r \;{}^{\wedge}\!/.\; s_2$ and $r \;{}^{\wedge}\!/.\; s_1 = \wedge \{ r_\alpha \in \mathbb{PR} \,|\, r_\alpha \cdot s_1 \geq r \}$ and $r \;{}^{\wedge}\!/.\; s_2 = \wedge \left\{ r'_\beta \in \mathbb{PR} \,\middle|\, r'_\beta \cdot s_2 \geq r \right\}$.

Let $r_\alpha \cdot s_1 \geq r$. If $s_1 \leq s_2$, then from the monotony of product we have $r_\alpha \cdot s_1 \leq r_\alpha \cdot s_2$, but $r_\alpha \cdot s_1 \geq r$, therefore $r_\alpha \cdot s_2 \geq r$. So each preradical r_α is one of preradicals r'_β, what implies $\wedge \{ r_\alpha \in \mathbb{PR} \,|\, r_\alpha \cdot s_1 \geq r \} \geq \wedge \left\{ r'_\beta \in \mathbb{PR} \,\middle|\, r'_\beta \cdot s_2 \geq r \right\}$, i.e. $r \;{}^{\wedge}\!/.\; s_1 \geq r \;{}^{\wedge}\!/.\; s_2$. □

The following fact is very useful for the further investigations.

Proposition 2.5. *Let* $r, s \in \mathbb{PR}$ *and* $r \leq s$. *Then for every* $t \in \mathbb{PR}$ *we have:*

$$r \leq t \cdot s \Leftrightarrow r \;{}^{\wedge}\!/.\; s \leq t.$$

Proof. By Lemma 2.1 there exists the left quotient $r \;{}^{\wedge}\!/.\; s$ and $r \;{}^{\wedge}\!/.\; s = \wedge \{ r_\alpha \in \mathbb{PR} \,|\, r_\alpha \cdot s \geq r \}$.

($\Rightarrow$) Let $t \cdot s \geq r$. Then t is one of preradicals r_α, therefore $t \geq \wedge \{ r_\alpha \in \mathbb{PR} \,|\, r_\alpha \cdot s \geq r \}$, i.e. $t \geq r \;{}^{\wedge}\!/.\; s$.

($\Leftarrow$) Let $r \;{}^{\wedge}\!/.\; s \leq t$. From the monotony of product $(r \;{}^{\wedge}\!/.\; s) \cdot s \leq t \cdot s$, but by the definition of the left quotient we have $(r \;{}^{\wedge}\!/.\; s) \cdot s \geq r$, therefore $t \cdot s \geq r$. □

In continuation we show some properties of the studied operation.

Proposition 2.6. *For every preradicals* $r, s \in \mathbb{PR}$ *we have:*

$$(r \cdot s) \;{}^{\wedge}\!/.\; s \leq r.$$

Proof. Since $r \cdot s \leq s$, then there exists the left quotient $(r \cdot s) \;{}^{\wedge}\!/.\; s$ and from Lemma 2.1 $(r \cdot s) \;{}^{\wedge}\!/.\; s = \wedge \{ t_\alpha \in \mathbb{PR} \,|\, t_\alpha \cdot s \geq r \cdot s \}$. Because $r \cdot s \geq r \cdot s$, the preradical r is one of preradicals t_α, therefore $r \geq \wedge \{ t_\alpha \in \mathbb{PR} \,|\, t_\alpha \cdot s \geq r \cdot s \}$, i.e. $r \geq (r \cdot s) \;{}^{\wedge}\!/.\; s$. □

Proposition 2.7. *Let* $r, s \in \mathbb{PR}$. *The following relations are true:*

1) $(r \;{}^{\wedge}\!/.\; s) \;{}^{\wedge}\!/.\; t = r \;{}^{\wedge}\!/.\; (t \cdot s)$ *for any preradical* t *with the property* $t \cdot s \geq r$;

2) $(r \cdot s) \;{}^{\wedge}\!/.\; t \leq r \cdot (s \;{}^{\wedge}\!/.\; t)$ *for any preradical* $t \geq s$.

Proof. 1) If $r \leq t \cdot s$, then there exists the left quotient $r \;{}^{\wedge}\!/.\; (t \cdot s)$. In this case, since $t \cdot s \leq s$ we have $r \leq s$, so there exists the left quotient $r \;{}^{\wedge}\!/.\; s$. Moreover, by Proposition 2.5 $t \cdot s \geq r \Leftrightarrow r \;{}^{\wedge}\!/.\; s \leq t$, which ensures the existence of the left quotient $(r \;{}^{\wedge}\!/.\; s) \;{}^{\wedge}\!/.\; t$. From Lemma 2.1 we have $r \;{}^{\wedge}\!/.\; (t \cdot s) = \wedge \{ r_\alpha \in \mathbb{PR} \,|\, r_\alpha \cdot (t \cdot s) \geq r \}$ and $(r \;{}^{\wedge}\!/.\; s) \;{}^{\wedge}\!/.\; t = \wedge \{ t_\beta \in \mathbb{PR} \,|\, t_\beta \cdot t \geq r \;{}^{\wedge}\!/.\; s \}$.

($\leq$) Let $r_\alpha \cdot (t \cdot s) \geq r$. Since $(r_\alpha \cdot t) \cdot s = r_\alpha \cdot (t \cdot s)$ we have $(r_\alpha \cdot t) \cdot s \geq r$, but $r \;{}^{\wedge}\!/.\; s$ is the least preradical with such property, so

$r_\alpha \cdot t \geq r \mathbin{{}^{\wedge}\!/.} s$. Hence each r_α is one of preradicals t_β, which implies that for any r_α we have $r_\alpha \geq \wedge \{ t_\beta \in \mathbb{PR} \mid t_\beta \cdot t \geq r \mathbin{{}^{\wedge}\!/.} s\}$ for every α, therefore $\wedge \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot (t \cdot s) \geq r\} \geq \wedge \{ t_\beta \in \mathbb{PR} \mid t_\beta \cdot t \geq r \mathbin{{}^{\wedge}\!/.} s\}$, i.e. $r \mathbin{{}^{\wedge}\!/.} (t \cdot s) \geq (r \mathbin{{}^{\wedge}\!/.} s) \mathbin{{}^{\wedge}\!/.} t$.

($\geq$) Let $t_\beta \cdot t \geq r \mathbin{{}^{\wedge}\!/.} s$. Using the associativity and the monotony of product of preradicals we obtain $t_\beta \cdot (t \cdot s) = (t_\beta \cdot t) \cdot s \geq (r \mathbin{{}^{\wedge}\!/.} s) \cdot s$, by the definition of the left quotient $(r \mathbin{{}^{\wedge}\!/.} s) \cdot s \geq r$, so $t_\beta \cdot (t \cdot s) \geq r$. This shows that each preradical t_β is one of preradicals r_α, therefore we have $\wedge \{ t_\beta \in \mathbb{PR} \mid t_\beta \cdot t \geq r \mathbin{{}^{\wedge}\!/.} s\} \geq \wedge \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot (t \cdot s) \geq r\}$, i.e $(r \mathbin{{}^{\wedge}\!/.} s) \mathbin{{}^{\wedge}\!/.} t \geq r \mathbin{{}^{\wedge}\!/.} (t \cdot s)$.

2) The relation $s \leq t$ ensures the existence of the left quotient $s \mathbin{{}^{\wedge}\!/.} t$ and since $r \cdot s \leq s$ we have $r \cdot s \leq t$, so there exists and the left quotient $(r \cdot s) \mathbin{{}^{\wedge}\!/.} t$.

By the definition of the left quotient $s \leq (s \mathbin{{}^{\wedge}\!/.} t) \cdot t$. Using the monotony and the property of associativity of product of preradicals it follows that $r \cdot s \leq r \cdot [(s \mathbin{{}^{\wedge}\!/.} t) \cdot t] = [r \cdot (s \mathbin{{}^{\wedge}\!/.} t)] \cdot t$. Applying Proposition 2.5 for relation $r \cdot s \leq [r \cdot (s \mathbin{{}^{\wedge}\!/.} t)] \cdot t$ we obtain $(r \cdot s) \mathbin{{}^{\wedge}\!/.} t \leq r \cdot (s \mathbin{{}^{\wedge}\!/.} t)$. □

Proposition 2.8. *Let* $r, s, t \in \mathbb{PR}$ *and* $r \leq s$. *Then the following relations hold:*

1) $(r \mathbin{{}^{\wedge}\!/.} t) \mathbin{{}^{\wedge}\!/.} (s \mathbin{{}^{\wedge}\!/.} t) \leq r \mathbin{{}^{\wedge}\!/.} s$ *or* $(r \mathbin{{}^{\wedge}\!/.} s) \cdot (s \mathbin{{}^{\wedge}\!/.} t) \geq r \mathbin{{}^{\wedge}\!/.} t$ *for any preradical* $t \geq s$*;*

2) $(r \cdot t) \mathbin{{}^{\wedge}\!/.} (s \cdot t) \leq r \mathbin{{}^{\wedge}\!/.} s$ *for any preradical* $t \in \mathbb{PR}$*.*

Proof. 1) The conditions $r \leq s$ and $s \leq t$ ensure the existence of the left quotients $r \mathbin{{}^{\wedge}\!/.} s$ and $s \mathbin{{}^{\wedge}\!/.} t$. In this case, since $r \leq s$ and $s \leq t$ we have $r \leq t$, so there exists the left quotient $r \mathbin{{}^{\wedge}\!/.} t$. Moreover, since $r \leq s$, from the monotony of the left quotient it follows that $r \mathbin{{}^{\wedge}\!/.} t \leq s \mathbin{{}^{\wedge}\!/.} t$, which ensures the existence of the left quotient $(r \mathbin{{}^{\wedge}\!/.} t) \mathbin{{}^{\wedge}\!/.} (s \mathbin{{}^{\wedge}\!/.} t)$.

From Proposition 2.5 the relations of this statement are equivalent.

By the definition of the left quotient $r \leq (r \mathbin{{}^{\wedge}\!/.} s) \cdot s$ and $s \leq (s \mathbin{{}^{\wedge}\!/.} t) \cdot t$. Therefore, using the monotony and the associativity of product we obtain $r \leq (r \mathbin{{}^{\wedge}\!/.} s) \cdot s \leq (r \mathbin{{}^{\wedge}\!/.} s) \cdot [(s \mathbin{{}^{\wedge}\!/.} t) \cdot t] = [(r \mathbin{{}^{\wedge}\!/.} s) \cdot (s \mathbin{{}^{\wedge}\!/.} t)] \cdot t$. Applying Proposition 2.5 for the relation $r \leq [(r \mathbin{{}^{\wedge}\!/.} s) \cdot (s \mathbin{{}^{\wedge}\!/.} t)] \cdot t$ we have $r \mathbin{{}^{\wedge}\!/.} t \leq (r \mathbin{{}^{\wedge}\!/.} s) \cdot (s \mathbin{{}^{\wedge}\!/.} t)$.

2) The condition $r \leq s$ ensures the existence of the left quotient $r \mathbin{{}^{\wedge}\!/.} s$. In this case, by the monotony of product we have $r \cdot t \leq s \cdot t$ for every $t \in \mathbb{PR}$, so there exists the left quotient $(r \cdot t) \mathbin{{}^{\wedge}\!/.} (s \cdot t)$.

From Proposition 2.5 the relation of this statement is equivalent to the relation $r \cdot t \leq (r \mathbin{{}^{\wedge}\!/.} s) \cdot (s \cdot t)$.

By the definition of the left quotient $r \leq (r \mathbin{{}^{\wedge}\!/.} s) \cdot s$, therefore applying the monotony and the associativity of product we obtain $r \cdot t \leq [(r \mathbin{{}^{\wedge}\!/.} s) \cdot s] \cdot t = (r \mathbin{{}^{\wedge}\!/.} s) \cdot (s \cdot t)$. □

Now we will discuss the question on the relations beetween the left quotient with respect to meet and the lattice operations of $\mathbb{PR}$.

Proposition 2.9. *(The left distributivity of the left quotient $r \,\gamma.\, s$ relative to join) Let $s \in \mathbb{PR}$. Then for any family of preradicals $\{r_\alpha \in \mathbb{PR} \,|\, r_\alpha \le s,\ \alpha \in \mathfrak{A}\}$ the following relation holds:*

$$\left(\bigvee_{\alpha\in\mathfrak{A}} r_\alpha\right) \gamma.\ s = \bigvee_{\alpha\in\mathfrak{A}} (r_\alpha \,\gamma.\, s).$$

Proof. The relations $r_\alpha \le s,\ \alpha \in \mathfrak{A}$ ensure the existence of the left quotients $r_\alpha \,\gamma.\, s,\ \alpha \in \mathfrak{A}$. But in this case $\bigvee_{\alpha\in\mathfrak{A}} r_\alpha \le s$, so there exists the left quotient $\left(\bigvee_{\alpha\in\mathfrak{A}} r_\alpha\right) \gamma.\ s$.

($\le$) By the definition of the left quotient $r_\alpha \le (r_\alpha \,\gamma.\, s) \cdot s$ for all $\alpha \in \mathfrak{A}$, which implies that $\bigvee_{\alpha\in\mathfrak{A}} r_\alpha \le \bigvee_{\alpha\in\mathfrak{A}} [(r_\alpha \,\gamma.\, s) \cdot s]$. Using the distributivity of product of preradicals relative to join $\bigvee_{\alpha\in\mathfrak{A}} r_\alpha \le \left[\bigvee_{\alpha\in\mathfrak{A}} (r_\alpha \,\gamma.\, s)\right] \cdot s$. Applying Proposition 2.5 we obtain $\left(\bigvee_{\alpha\in\mathfrak{A}} r_\alpha\right) \gamma.\ s \le \bigvee_{\alpha\in\mathfrak{A}} (r_\alpha \,\gamma.\, s)$.

($\ge$) From Lemma 2.1 $\left(\bigvee_{\alpha\in\mathfrak{A}} r_\alpha\right) \gamma.\ s = \wedge \left\{ t_\beta \in \mathbb{PR} \,|\, t_\beta \cdot s \ge \bigvee_{\alpha\in\mathfrak{A}} r_\alpha \right\}$ and $r_\alpha \,\gamma.\, s = \wedge \{ r'_\gamma \in \mathbb{PR} \,|\, r'_\gamma \cdot s \ge r_\alpha \}$.

Let $t_\beta \cdot s \ge \bigvee_{\alpha\in\mathfrak{A}} r_\alpha$. Since $\bigvee_{\alpha\in\mathfrak{A}} r_\alpha \ge r_\alpha$ for every $\alpha \in \mathfrak{A}$ we have $t_\beta \cdot s \ge r_\alpha$, hence each preradical t_β is one of preradicals r'_γ. Therefore $\wedge \left\{ t_\beta \in \mathbb{PR} \,|\, t_\beta \cdot s \ge \bigvee_{\alpha\in\mathfrak{A}} r_\alpha \right\} \ge \wedge \{ r'_\gamma \in \mathbb{PR} \,|\, r'_\gamma \cdot s \ge r_\alpha \}$ for every $\alpha \in \mathfrak{A}$, i.e. $\left(\bigvee_{\alpha\in\mathfrak{A}} r_\alpha\right) \gamma.\ s \ge r_\alpha \,\gamma.\, s$ for every $\alpha \in \mathfrak{A}$, which implies that $\left(\bigvee_{\alpha\in\mathfrak{A}} r_\alpha\right) \gamma.\ s \ge \bigvee_{\alpha\in\mathfrak{A}} (r_\alpha \,\gamma.\, s)$. □

Proposition 2.10. *In the class $\mathbb{PR}$ the following relations are true:*

1) $\left(\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha\right) \gamma.\ s \le \bigwedge_{\alpha\in\mathfrak{A}} (r_\alpha \,\gamma.\, s)$, *when $r_\alpha \le s$ for any $\alpha \in \mathfrak{A}$;*

2) $r \,\gamma.\, \left(\bigwedge_{\alpha\in\mathfrak{A}} s_\alpha\right) \ge \bigvee_{\alpha\in\mathfrak{A}} (r \,\gamma.\, s_\alpha)$, *when $r \le s_\alpha$ for any $\alpha \in \mathfrak{A}$;*

3) $r \,\gamma.\, \left(\bigvee_{\alpha\in\mathfrak{A}} s_\alpha\right) \le \bigwedge_{\alpha\in\mathfrak{A}} (r \,\gamma.\, s_\alpha)$, *when $r \le s_\alpha$ for any $\alpha \in \mathfrak{A}$.*

Proof. 1) The conditions $r_\alpha \le s$, $\alpha \in \mathfrak{A}$ ensure the existence of the left quotients $r_\alpha \mathbin{/\!\!\cdot} s$, $\alpha \in \mathfrak{A}$. But in this case $\bigwedge\limits_{\alpha\in\mathfrak{A}} r_\alpha \leq s$, so there exists the left quotient $\left(\bigwedge\limits_{\alpha\in\mathfrak{A}} r_\alpha\right) \mathbin{/\!\!\cdot} s$.

By the definition of the left quotient we have $r_\alpha \leq (r_\alpha \mathbin{/\!\!\cdot} s) \cdot s$ for any $\alpha \in \mathfrak{A}$, which implies that $\bigwedge\limits_{\alpha\in\mathfrak{A}} r_\alpha \leq \bigwedge\limits_{\alpha\in\mathfrak{A}} [(r_\alpha \mathbin{/\!\!\cdot} s) \cdot s]$. Applying the distributivity of product of preradicals relative to meet it follows that $\bigwedge\limits_{\alpha\in\mathfrak{A}} r_\alpha \leq \left[\bigwedge\limits_{\alpha\in\mathfrak{A}} (r_\alpha \mathbin{/\!\!\cdot} s)\right] \cdot s$ and from Proposition 2.5 we obtain $\left(\bigwedge\limits_{\alpha\in\mathfrak{A}} r_\alpha\right) \mathbin{/\!\!\cdot} s \leq \bigwedge\limits_{\alpha\in\mathfrak{A}} (r_\alpha \mathbin{/\!\!\cdot} s)$.

2) The conditions $r \leq s_\alpha$, $\alpha \in \mathfrak{A}$ ensure the existence of the left quotients $r \mathbin{/\!\!\cdot} s_\alpha$, $\alpha \in \mathfrak{A}$. But in this case $r \leq \bigwedge\limits_{\alpha\in\mathfrak{A}} s_\alpha$, which implies the existence of the left quotient $r \mathbin{/\!\!\cdot} \left(\bigwedge\limits_{\alpha\in\mathfrak{A}} s_\alpha\right)$.

For any $\alpha \in \mathfrak{A}$ we have $\bigwedge\limits_{\alpha\in\mathfrak{A}} s_\alpha \leq s_\alpha$. Using Proposition 2.4 we obtain $r \mathbin{/\!\!\cdot} \left(\bigwedge\limits_{\alpha\in\mathfrak{A}} s_\alpha\right) \geq r \mathbin{/\!\!\cdot} s_\alpha$ for all $\alpha \in \mathfrak{A}$, therefore $r \mathbin{/\!\!\cdot} \left(\bigwedge\limits_{\alpha\in\mathfrak{A}} s_\alpha\right) \geq \bigvee\limits_{\alpha\in\mathfrak{A}} (r \mathbin{/\!\!\cdot} s_\alpha)$.

3) The conditions $r \leq s_\alpha$, $\alpha \in \mathfrak{A}$ ensure the existence of the left quotients $r \mathbin{/\!\!\cdot} s_\alpha$, $\alpha \in \mathfrak{A}$. Moreover, in this case $r \leq \bigvee\limits_{\alpha\in\mathfrak{A}} s_\alpha$, so there exists the left quotient $r \mathbin{/\!\!\cdot} \left(\bigvee\limits_{\alpha\in\mathfrak{A}} s_\alpha\right)$.

For any $\alpha \in \mathfrak{A}$ we have $\bigvee\limits_{\alpha\in\mathfrak{A}} s_\alpha \geq s_\alpha$. Using Proposition 2.4 we obtain $r \mathbin{/\!\!\cdot} \left(\bigvee\limits_{\alpha\in\mathfrak{A}} s_\alpha\right) \leq r \mathbin{/\!\!\cdot} s_\alpha$ for all $\alpha \in \mathfrak{A}$, therefore $r \mathbin{/\!\!\cdot} \left(\bigvee\limits_{\alpha\in\mathfrak{A}} s_\alpha\right) \leq \bigwedge\limits_{\alpha\in\mathfrak{A}} (r \mathbin{/\!\!\cdot} s_\alpha)$. □

Now we will consider some particular cases of the left quotient $r \mathbin{/\!\!\cdot} s$.

Proposition 2.11. *Let* $r, s \in \mathbb{PR}$. *Then:*

1) $r \mathbin{/\!\!\cdot} r = e\,(r)$ *(see Definition 1.1);*
2) $r \mathbin{/\!\!\cdot} 1 = r$;
3) $0 \mathbin{/\!\!\cdot} s = 0$.

Proof. Using the definition of the left quotient we obtain:

1) $r \mathbin{/\!\!\cdot} r = \wedge\,\{r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot r \geq r\} = \wedge\,\{r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot r = r\} = e\,(r)$;
2) $r \mathbin{/\!\!\cdot} 1 = \wedge\,\{r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot 1 \geq r\} = \wedge\,\{r_\alpha \in \mathbb{PR} \mid r_\alpha \geq r\} = r$;
3) $0 \mathbin{/\!\!\cdot} s = \wedge\,\{r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s \geq 0\} = \wedge\,\{r_\alpha \mid r_\alpha \in \mathbb{PR}\} = 0$. □

By Proposition 2.11 we have the following particular cases:

(1) $0 \mathbin{{}^{\wedge}\!/\!.} 0 = 0\,$; (2) $1 \mathbin{{}^{\wedge}\!/\!.} 1 = 1$.

Applying to the relation $r \leq s \leq 1$ the statement of Proposition 2.4, we obtain $r \mathbin{{}^{\wedge}\!/\!.} r \geq r \mathbin{{}^{\wedge}\!/\!.} s \geq r \mathbin{{}^{\wedge}\!/\!.} 1$, hence $r \leq r \mathbin{{}^{\wedge}\!/\!.} s \leq e\,(r)$.

Moreover, the distributivity of product of preradicals relative to meet implies $e\,(r) \cdot r = \left(\underset{r_\alpha \cdot r = r}{\wedge} r_\alpha \right) \cdot r = \underset{r_\alpha \cdot r = r}{\wedge} (r_\alpha \cdot r) = r$ for every $r \in \mathbb{PR}$.

The following statement shows some properties of equalizer.

Proposition 2.12. *Let* $r, s \in \mathbb{PR}$ *and* $r \leq s$. *Then:*

1) $e\,(r) \cdot (r \mathbin{{}^{\wedge}\!/\!.} s) = r \mathbin{{}^{\wedge}\!/\!.} s$;
2) $(r \mathbin{{}^{\wedge}\!/\!.} s) \cdot e\,(s) = r \mathbin{{}^{\wedge}\!/\!.} s$;
3) $(r \mathbin{{}^{\wedge}\!/\!.} s) \mathbin{{}^{\wedge}\!/\!.} e\,(s) = r \mathbin{{}^{\wedge}\!/\!.} s$.

Proof. The condition $r \leq s$ ensures the existence of the left quotient $r \mathbin{{}^{\wedge}\!/\!.} s$.

1) $e\,(r) \cdot (r \mathbin{{}^{\wedge}\!/\!.} s) = (r \mathbin{{}^{\wedge}\!/\!.} r) \cdot (r \mathbin{{}^{\wedge}\!/\!.} s)$. Applying Proposition 2.8(1) we obtain $(r \mathbin{{}^{\wedge}\!/\!.} r) \cdot (r \mathbin{{}^{\wedge}\!/\!.} s) \geq r \mathbin{{}^{\wedge}\!/\!.} s$, but $(r \mathbin{{}^{\wedge}\!/\!.} r) \cdot (r \mathbin{{}^{\wedge}\!/\!.} s) \leq r \mathbin{{}^{\wedge}\!/\!.} s$, so $e\,(r) \cdot (r \mathbin{{}^{\wedge}\!/\!.} s) = r \mathbin{{}^{\wedge}\!/\!.} s$.

2) $(r \mathbin{{}^{\wedge}\!/\!.} s) \cdot e\,(s) = (r \mathbin{{}^{\wedge}\!/\!.} s) \cdot (s \mathbin{{}^{\wedge}\!/\!.} s)$. Applying Proposition 2.8(1) we obtain $(r \mathbin{{}^{\wedge}\!/\!.} s) \cdot (s \mathbin{{}^{\wedge}\!/\!.} s) \geq r \mathbin{{}^{\wedge}\!/\!.} s$, but $(r \mathbin{{}^{\wedge}\!/\!.} s) \cdot (s \mathbin{{}^{\wedge}\!/\!.} s) \leq r \mathbin{{}^{\wedge}\!/\!.} s$, so $(r \mathbin{{}^{\wedge}\!/\!.} s) \cdot e\,(s) = r \mathbin{{}^{\wedge}\!/\!.} s$.

3) $(r \mathbin{{}^{\wedge}\!/\!.} s) \mathbin{{}^{\wedge}\!/\!.} e\,(s) = (r \mathbin{{}^{\wedge}\!/\!.} s) \mathbin{{}^{\wedge}\!/\!.} (s \mathbin{{}^{\wedge}\!/\!.} s)$. Using Proposition 2.8(1) we obtain $(r \mathbin{{}^{\wedge}\!/\!.} s) \mathbin{{}^{\wedge}\!/\!.} (s \mathbin{{}^{\wedge}\!/\!.} s) \leq r \mathbin{{}^{\wedge}\!/\!.} s$, but from Lemma 2.2 $(r \mathbin{{}^{\wedge}\!/\!.} s) \mathbin{{}^{\wedge}\!/\!.} (s \mathbin{{}^{\wedge}\!/\!.} s) \geq r \mathbin{{}^{\wedge}\!/\!.} s$, so we have $(r \mathbin{{}^{\wedge}\!/\!.} s) \mathbin{{}^{\wedge}\!/\!.} e\,(s) = r \mathbin{{}^{\wedge}\!/\!.} s$.

□

We will now consider the case of idempotent preradicals.

Remark 2.13. ([5]) *For every preradical* $r \in \mathbb{PR}$ *we have* $e\,(r)$ *is an idempotent preradical.*

Proof. $e\,(r) \cdot e\,(r) = (r \mathbin{{}^{\wedge}\!/\!.} r) \cdot (r \mathbin{{}^{\wedge}\!/\!.} r)$. From Proposition 2.8(1) we have $(r \mathbin{{}^{\wedge}\!/\!.} r) \cdot (r \mathbin{{}^{\wedge}\!/\!.} r) \geq r \mathbin{{}^{\wedge}\!/\!.} r$, but $(r \mathbin{{}^{\wedge}\!/\!.} r) \cdot (r \mathbin{{}^{\wedge}\!/\!.} r) \leq r \mathbin{{}^{\wedge}\!/\!.} r$, hence $e\,(r) \cdot e\,(r) = e\,(r)$, i.e. $e\,(r)$ is an idempotent preradical. □

Proposition 2.14. ([5]) *The preradical* $r \in \mathbb{PR}$ *is idempotent if and only if* $e\,(r) = r$.

Proof. ($\Rightarrow$) By Definition 1.2 $e\,(r) = \wedge \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot r = r \}$. If r is an idempotent preradical, i.e. $r \cdot r = r$, then r is one of preradicals r_α. Therefore $r \geq \wedge \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot r = r \}$, i.e. $r \geq e\,(r)$, but $e\,(r) \geq r$, so $e\,(r) = r$.

($\Leftarrow$) Let $e\,(r) = r$. Then $r \cdot r = e\,(r) \cdot r$. Since $e\,(r) \cdot r = r$, so $r \cdot r = r$, i.e. r is idempotent. □

Moreover, since $r \le r \mathbin{\gamma\!.} s \le e(r)$, if r is idempotent, then $r \mathbin{\gamma\!.} s = r$.

Proposition 2.15. *Let* $r, s \in \mathbb{PR}$ *and* s *be an idempotent preradical. Then:*

1) $r \mathbin{\gamma\!.} s \le s$ *with* $r \le s$*;*
2) $(r \mathbin{\gamma\!.} s) \cdot s = r \mathbin{\gamma\!.} s$ *with* $r \le s$*;*
3) $(r \mathbin{\gamma\!.} s) \mathbin{\gamma\!.} s = r \mathbin{\gamma\!.} s$ *with* $r \le s$*;*
4) $(r \cdot s) \mathbin{\gamma\!.} s = r \cdot s$.

Proof. 1) The condition $r \le s$ ensures the existence of the left quotient $r \mathbin{\gamma\!.} s$.

Let $r \le s$. From the monotony of the left quotient $r \mathbin{\gamma\!.} s \le s \mathbin{\gamma\!.} s$. If s is idempotent, then $r \mathbin{\gamma\!.} s \le s$.

2) The condition $r \le s$ ensures the existence of the left quotient $r \mathbin{\gamma\!.} s$.

If s is an idempotent preradical, then $(r \mathbin{\gamma\!.} s) \cdot s = (r \mathbin{\gamma\!.} s) \cdot e(s)$. By Proposition 2.12 we obtain $(r \mathbin{\gamma\!.} s) \cdot s = r \mathbin{\gamma\!.} s$.

3) The condition $r \le s$ ensures the existence of the left quotient $r \mathbin{\gamma\!.} s$. Moreover, from 1) $r \mathbin{\gamma\!.} s \le s$, which implies the existence of the left quotient $(r \mathbin{\gamma\!.} s) \mathbin{\gamma\!.} s$.

From Proposition 2.7(1) $(r \mathbin{\gamma\!.} s) \mathbin{\gamma\!.} s = r \mathbin{\gamma\!.} (s \cdot s)$. If s is an idempotent preradical, then $(r \mathbin{\gamma\!.} s) \mathbin{\gamma\!.} s = r \mathbin{\gamma\!.} s$.

4) From Proposition 2.7(2) $(r \cdot s) \mathbin{\gamma\!.} s \le r \cdot (s \mathbin{\gamma\!.} s)$. If s is an idempotent preradical, then $(r \cdot s) \mathbin{\gamma\!.} s \le r \cdot s$. But from Lemma 2.2 $(r \cdot s) \mathbin{\gamma\!.} s \ge r \cdot s$, therefore $(r \cdot s) \mathbin{\gamma\!.} s = r \cdot s$. □

The next two statements show when the cancellation properties for the left quotient hold (see Proposition 2.6).

Proposition 2.16. *Let* $r, s \in \mathbb{PR}$*. The following conditions are equivalent:*

1) $r = (r \cdot s) \mathbin{\gamma\!.} s$*;*
2) $r = t \mathbin{\gamma\!.} s$ *for some preradical* $t \le s$.

Proof. The condition $t \le s$ ensures the existence of the left quotient $t \mathbin{\gamma\!.} s$.

1) $\Rightarrow$ 2) If $r = (r \cdot s) \mathbin{\gamma\!.} s$, then $r = t \mathbin{\gamma\!.} s$ with $t = r \cdot s$.

2) $\Rightarrow$ 1) Let $r = t \mathbin{\gamma\!.} s$ for some preradical $t \le s$. By the definition of the left quotient $(t \mathbin{\gamma\!.} s) \cdot s \ge t$. Using the monotony of the left quotient we obtain $[(t \mathbin{\gamma\!.} s) \cdot s] \mathbin{\gamma\!.} s \ge t \mathbin{\gamma\!.} s$. But from Proposition 2.6 $[(t \mathbin{\gamma\!.} s) \cdot s] \mathbin{\gamma\!.} s \le t \mathbin{\gamma\!.} s$, thus $[(t \mathbin{\gamma\!.} s) \cdot s] \mathbin{\gamma\!.} s = t \mathbin{\gamma\!.} s$. Since $t \mathbin{\gamma\!.} s = r$, we have $(r \cdot s) \mathbin{\gamma\!.} s = r$. □

Proposition 2.17. *Let* $r, s \in \mathbb{PR}$*. The following conditions are equivalent:*

1) $r = (r \mathbin{\gamma\!.} s) \cdot s$ *with* $r \le s$*;*
2) $r = t \cdot s$ *for some preradical* $t \in \mathbb{PR}$.

Proof. The condition $r \le s$ ensures the existence of the left quotient $r \,{}^{\wedge}\!/.\, s$.

1) $\Rightarrow$ 2) If $r = (r \,{}^{\wedge}\!/.\, s) \cdot s$, then $r = t \cdot s$ with $t = r \,{}^{\wedge}\!/.\, s$.

2) $\Rightarrow$ 1) Let $r = t \cdot s$ for some preradical $t \in \mathbb{PR}$. By Proposition 2.6 $(t \cdot s) \,{}^{\wedge}\!/.\, s \le t$. Applying the monotony of product we obtain $[(t \cdot s) \,{}^{\wedge}\!/.\, s] \cdot s \le t \cdot s$. But from the definition of the left quotient $[(t \cdot s) \,{}^{\wedge}\!/.\, s] \cdot s \ge t \cdot s$, therefore $[(t \cdot s) \,{}^{\wedge}\!/.\, s] \cdot s = t \cdot s$. Since $t \cdot s = r$, we have $(r \,{}^{\wedge}\!/.\, s) \cdot s = r$. $\square$

Now we will study the behaviour of the left quotient $r \,{}^{\wedge}\!/.\, s$ in the case of such types of preradicals as coprime, $\vee$-coprime and coirreducible (see Section 1).

Proposition 2.18. *If r is a coprime preradical, then the preradical $r \,{}^{\wedge}\!/.\, s$ is coprime for any preradical $s \ge r$.*

Proof. The condition $r \le s$ ensures the existence of the left quotient $r \,{}^{\wedge}\!/.\, s$.

Let the preradical $r \ne 0$ be coprime and $t_1 \# t_2 \ge r \,{}^{\wedge}\!/.\, s$ for some preradicals $t_1, t_2 \in \mathbb{PR}$. Using Proposition 2.5 we obtain $r \le (t_1 \# t_2) \cdot s$. From Lemma 1.3(2) $(t_1 \# t_2) \cdot s \le (t_1 \cdot s) \# (t_2 \cdot s)$, so $r \le (t_1 \cdot s) \# (t_2 \cdot s)$. Since r is coprime, it follows that $r \le t_1 \cdot s$ or $r \le t_2 \cdot s$. Applying Proposition 2.5 we obtain $r \,{}^{\wedge}\!/.\, s \le t_1$ or $r \,{}^{\wedge}\!/.\, s \le t_2$. So for every $t_1, t_2 \in \mathbb{PR}$ with $t_1 \# t_2 \ge r \,{}^{\wedge}\!/.\, s$ we have $t_1 \ge r \,{}^{\wedge}\!/.\, s$ or $t_2 \ge r \,{}^{\wedge}\!/.\, s$, which means that the preradical $r \,{}^{\wedge}\!/.\, s$ is coprime. $\square$

Proposition 2.19. *If the preradical r is $\vee$-coprime, then the preradical $r \,{}^{\wedge}\!/.\, s$ is $\vee$-coprime for any preradical $s \ge r$.*

Proof. The condition $r \le s$ ensures the existence of the left quotient $r \,{}^{\wedge}\!/.\, s$.

Let r be $\vee$-coprime and $t_1 \vee t_2 \ge r \,{}^{\wedge}\!/.\, s$, for some preradicals $t_1, t_2 \in \mathbb{PR}$. Applying Proposition 2.5 we obtain $r \le (t_1 \vee t_2) \cdot s$. From the distributivity of product of preradicals relative to join $r \le (t_1 \cdot s) \vee (t_2 \cdot s)$. Since r is $\vee$-coprime it follows that $r \le t_1 \cdot s$ or $r \le t_2 \cdot s$. From Proposition 2.5 we obtain $r \,{}^{\wedge}\!/.\, s \le t_1$ or $r \,{}^{\wedge}\!/.\, s \le t_2$. So for every preradicals $t_1, t_2 \in \mathbb{PR}$ with $t_1 \vee t_2 \ge r \,{}^{\wedge}\!/.\, s$ we have $t_1 \ge r \,{}^{\wedge}\!/.\, s$ or $t_2 \ge r \,{}^{\wedge}\!/.\, s$, which means that the preradical $r \,{}^{\wedge}\!/.\, s$ is $\vee$-coprime. $\square$

Proposition 2.20. *Let $r, s \in \mathbb{PR}$ and $r = t \cdot s$ for some preradical $t \in \mathbb{PR}$. If the preradical r is coirreducible, then the preradical $r \,{}^{\wedge}\!/.\, s$ is coirreducible.*

Proof. By the condition $r = t \cdot s$ we have $r \le s$, which ensures the existence of the left quotient $r \,{}^{\wedge}\!/.\, s$.

Let r be coirreducible and $r \,{}^{\wedge}\!/.\, s = t_1 \vee t_2$ for some preradicals $t_1, t_2 \in \mathbb{PR}$. If $r = t \cdot s$ for some preradical t, then by Proposition 2.17 $r = (r \,{}^{\wedge}\!/.\, s) \cdot s$, thus $r = (t_1 \vee t_2) \cdot s$. From the distributivity of product of preradicals relative to join $r = (t_1 \cdot s) \vee (t_2 \cdot s)$. Since r is coirreducible it follows that $t_1 \cdot s = r$ or $t_2 \cdot s = r$.

If $t_1 \cdot s = r$, then from Proposition 2.5 we have $t_1 \geq r \,{}^{\wedge}\!/_{\cdot}\, s$. But $t_1 \leq t_1 \vee t_2 = r \,{}^{\wedge}\!/_{\cdot}\, s$, therefore $t_1 = r \,{}^{\wedge}\!/_{\cdot}\, s$.

If $t_2 \cdot s = r$, then similarly we obtain $t_2 = r \,{}^{\wedge}\!/_{\cdot}\, s$.

So for every preradicals $t_1, t_2 \in \mathbb{PR}$ with $t_1 \vee t_2 = r \,{}^{\wedge}\!/_{\cdot}\, s$ we have $t_1 = r \,{}^{\wedge}\!/_{\cdot}\, s$ or $t_2 = r \,{}^{\wedge}\!/_{\cdot}\, s$, which means that the preradical $r \,{}^{\wedge}\!/_{\cdot}\, s$ is coirreducible. □

Moreover, if the preradical r is coprime, then its equalizer $e\,(r)$ is coprime ([7]).

The operation of the left quotient with respect to meet implies some order relations between the associated preradicals.

Corollary 2.21. 1) *For every preradicals $r, s \in \mathbb{PR}$ with $r \leq s$ the following relations hold:*

$$r \cdot s \leq (r \cdot s) \,{}^{\wedge}\!/_{\cdot}\, s \leq r \leq (r \,{}^{\wedge}\!/_{\cdot}\, s) \cdot s \leq r \,{}^{\wedge}\!/_{\cdot}\, s;$$

2) *If the preradical s is idempotent, then*

$$r \cdot s = (r \cdot s) \,{}^{\wedge}\!/_{\cdot}\, s \leq r \leq (r \,{}^{\wedge}\!/_{\cdot}\, s) \cdot s = r \,{}^{\wedge}\!/_{\cdot}\, s \leq s$$

for every preradical $r \leq s$. □

3 Left coquotient with respect to join

In this section the similar questions are discussed as in the preceding one for the inverse operation of coproduct with respect to join in the class of preradicals $\mathbb{PR}$ of category R-Mod.

Definition 3.1. *Let $r, s \in \mathbb{PR}$. The* left coquotient with respect to join of r by s *is defined as the greatest preradical among $r_\alpha \in \mathbb{PR}$ with the property $r_\alpha \# s \leq r$. We denote this preradical by $r \,{}^{\vee}\!/_{\#}\, s$.*

We will call r the *numerator* and s the *denominator* of the left coquotient $r \,{}^{\vee}\!/_{\#}\, s$.

The following statement is the answer to the question on the existence of the left coquotient with respect to join.

Lemma 3.1. *Let $r, s \in \mathbb{PR}$. The left coquotient $r \,{}^{\vee}\!/_{\#}\, s$ of r by s with respect to join exists if and only if $r \geq s$ and it can be presented in the form $r \,{}^{\vee}\!/_{\#}\, s = \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \leq r \}$.*

Proof. ($\Rightarrow$) Let there exists the left coquotient $r \,{}^{\vee}\!/_{\#}\, s$. Then $\exists r_\alpha \in \mathbb{PR}$ such that $r_\alpha \# s \leq r$. Since $0 \leq r_\alpha$, from the monotony of coproduct of preradicals we obtain $0 \# s \leq r_\alpha \# s$, i.e. $s \leq r_\alpha \# s$, therefore $s \leq r$.

($\Leftarrow$) Let $r \geq s$. Then $0 \# s = s \leq r$, therefore the family of preradicals $\{ r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \leq r \}$ is not empty. So we can consider the preradical $\vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \leq r \}$. Using the distributivity of coproduct relative to join of preradicals we have $\left(\bigvee_{r_\alpha \# s \leq r} r_\alpha \right) \# s = \bigvee_{r_\alpha \# s \leq r} (r_\alpha \# s)$, but since $r_\alpha \# s \leq r$ for every preradical r_α it follows that $\bigvee_{r_\alpha \# s \leq r} (r_\alpha \# s) \leq r$, i.e. $\left(\bigvee_{r_\alpha \# s \leq r} r_\alpha \right) \# s \leq r$. By the construction of this preradical is clear that it is the greatest preradical of $\mathbb{PR}$ with the property $r_\alpha \# s \leq r$. Therefore $\vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \leq r \} = r \,{}^{\vee}\!\!/_{\#}\, s$. □

Moreover, by the proof of Lemma 3.1 we have that $(r \,{}^{\vee}\!\!/_{\#}\, s) \# s \leq r$.

Lemma 3.2. *For every $r, s \in \mathbb{PR}$ with $r \geq s$ we have $r \,{}^{\vee}\!\!/_{\#}\, s \leq r$.*

Proof. The condition $r \geq s$ ensures the existence of the left coquotient $r \,{}^{\vee}\!\!/_{\#}\, s$. Since $r \,{}^{\vee}\!\!/_{\#}\, s \leq (r \,{}^{\vee}\!\!/_{\#}\, s) \# s$, by the definition of the left coquotient we obtain that $r \,{}^{\vee}\!\!/_{\#}\, s \leq r$. □

Now we show the behaviour of the left coquotient relative to the partial order $(\leq)$ of the class $\mathbb{PR}$.

Proposition 3.3. *(Monotony in the numerator) Let $r_1, r_2 \in \mathbb{PR}$ and $r_1 \leq r_2$. Then for every preradical $s \leq r_1$ we have $r_1 \,{}^{\vee}\!\!/_{\#}\, s \leq r_2 \,{}^{\vee}\!\!/_{\#}\, s$.*

Proof. Since $s \leq r_1 \leq r_2$, from Lemma 3.1 there exist the left coquotients $r_1 \,{}^{\vee}\!\!/_{\#}\, s$, $r_2 \,{}^{\vee}\!\!/_{\#}\, s$ and $r_1 \,{}^{\vee}\!\!/_{\#}\, s = \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \leq r_1 \}$, $r_2 \,{}^{\vee}\!\!/_{\#}\, s = \vee \left\{ r'_\beta \in \mathbb{PR} \,\middle|\, r'_\beta \# s \leq r_2 \right\}$.

The relations $r_1 \leq r_2$ and $r_\alpha \# s \leq r_1$ imply $r_\alpha \# s \leq r_2$, so each r_α is one of preradicals r'_β, from where it follows that $\vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \leq r_1 \} \leq \vee \left\{ r'_\beta \in \mathbb{PR} \,\middle|\, r'_\beta \# s \leq r_2 \right\}$, so $r_1 \,{}^{\vee}\!\!/_{\#}\, s \leq r_2 \,{}^{\vee}\!\!/_{\#}\, s$. □

Proposition 3.4. *(Antimonotony in the denominator) Let $s_1, s_2 \in \mathbb{PR}$ and $s_1 \leq s_2$. Then for every preradical $r \geq s_2$ we have $r \,{}^{\vee}\!\!/_{\#}\, s_1 \geq r \,{}^{\vee}\!\!/_{\#}\, s_2$.*

Proof. Since $r \geq s_2 \geq s_1$ from Lemma 3.1 there exist the left coquotient $r \,{}^{\vee}\!\!/_{\#}\, s_1$, $r \,{}^{\vee}\!\!/_{\#}\, s_2$ and $r \,{}^{\vee}\!\!/_{\#}\, s_1 = \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \# s_1 \leq r \}$ and $r \,{}^{\vee}\!\!/_{\#}\, s_2 = \vee \left\{ r'_\beta \in \mathbb{PR} \,\middle|\, r'_\beta \# s_2 \leq r \right\}$.

Let $s_1 \leq s_2$, from the monotony of coproduct of preradicals we have $r'_\beta \# s_1 \leq r'_\beta \# s_2$, but if $r'_\beta \# s_2 \leq r$, then $r'_\beta \# s_1 \leq r$. Therefore each preradical r'_β is one of preradicals r_α, which implies that $\vee \left\{ r'_\beta \in \mathbb{PR} \,\middle|\, r'_\beta \# s_2 \leq r \right\} \leq \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \# s_1 \leq r \}$, i.e. $r \,{}^{\vee}\!\!/_{\#}\, s_2 \leq r \,{}^{\vee}\!\!/_{\#}\, s_1$. □

The next statement is useful for applications.

Proposition 3.5. *Let* $r, s \in \mathbb{PR}$ *and* $r \geq s$. *Then for every preradical* $t \in \mathbb{PR}$ *we have:*

$$r \geq t \# s \Leftrightarrow r \,{}^{\vee}\!/_{\#}\, s \geq t.$$

Proof. From Lemma 3.1 there exists the left coquotient $r \,{}^{\vee}\!/_{\#}\, s$ and $r \,{}^{\vee}\!/_{\#}\, s = \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \leq r \}$.

($\Rightarrow$) Let $t \# s \leq r$. Then t is one of preradicals r_α, hence $t \leq \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \leq r \} = r \,{}^{\vee}\!/_{\#}\, s$.

($\Leftarrow$) Let $t \leq r \,{}^{\vee}\!/_{\#}\, s$. From the monotony of coproduct $t \# s \leq (r \,{}^{\vee}\!/_{\#}\, s) \# s$. By the definition of the left coquotient we have $(r \,{}^{\vee}\!/_{\#}\, s) \# s \leq r$, thus $t \# s \leq r$. □

In the following statements some properties of the left coquotient are indicated.

Proposition 3.6. *For every preradicals* $r, s \in \mathbb{PR}$ *we have:*

$$(r \# s) \,{}^{\vee}\!/_{\#}\, s \geq r.$$

Proof. Since $r \# s \geq s$, from Lemma 3.1 there exists the left coquotient $(r \# s) \,{}^{\vee}\!/_{\#}\, s$ and $(r \# s) \,{}^{\vee}\!/_{\#}\, s = \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \leq r \# s \}$.

Since $r \# s \leq r \# s$, we have that the preradical r is one of preradicals r_α, therefore $r \leq \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \leq r \# s \}$, i.e. $r \leq (r \# s) \,{}^{\vee}\!/_{\#}\, s$. □

Proposition 3.7. *Let* $r, s, t \in \mathbb{PR}$. *The following relations are true:*

1) $(r \,{}^{\vee}\!/_{\#}\, s) \,{}^{\vee}\!/_{\#}\, t = r \,{}^{\vee}\!/_{\#}\, (t \# s)$ *with* $r \geq t \# s$;

2) $(r \# s) \,{}^{\vee}\!/_{\#}\, t \geq r \# (s \,{}^{\vee}\!/_{\#}\, t)$ *with* $s \geq t$.

Proof. 1) If $r \geq t \# s$, then there exists the left coquotient $r \,{}^{\vee}\!/_{\#}\, (t \# s)$. In this case, since $t \# s \geq s$ we have $r \geq s$, so there exists the left coquotient $r \,{}^{\vee}\!/_{\#}\, s$. Moreover, by Proposition 3.5 $t \# s \leq r \Leftrightarrow r \,{}^{\vee}\!/_{\#}\, s \geq t$, which ensures the existence of the left coquotient $(r \,{}^{\vee}\!/_{\#}\, s) \,{}^{\vee}\!/_{\#}\, t$. From Lemma 3.1 we have $r \,{}^{\vee}\!/_{\#}\, s = \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \leq r \}$, $r \,{}^{\vee}\!/_{\#}\, (t \# s) = \vee \{ s_\beta \in \mathbb{PR} \mid s_\beta \# (t \# s) \leq r \}$ and $(r \,{}^{\vee}\!/_{\#}\, s) \,{}^{\vee}\!/_{\#}\, t = \vee \{ t_\gamma \in \mathbb{PR} \mid t_\gamma \# t \leq r \,{}^{\vee}\!/_{\#}\, s \}$.

($\leq$) Let $t_\gamma \# t \leq r \,{}^{\vee}\!/_{\#}\, s$. Using the monotony of coproduct of preradicals we obtain $(t_\gamma \# t) \# s \leq (r \,{}^{\vee}\!/_{\#}\, s) \# s$, but from the definition of the left coquotient $(r \,{}^{\vee}\!/_{\#}\, s) \# s \leq r$, so $(t_\gamma \# t) \# s \leq r$. By the associativity of coproduct of preradicals we have $t_\gamma \# (t \# s) \leq r$, which means that each preradical t_γ is one of preradicals s_β, therefore $\vee \{ t_\gamma \in \mathbb{PR} \mid t_\gamma \# t \leq r \,{}^{\vee}\!/_{\#}\, s \} \leq \vee \{ s_\beta \in \mathbb{PR} \mid s_\beta \# (t \# s) \leq r \}$, i.e $(r \,{}^{\vee}\!/_{\#}\, s) \,{}^{\vee}\!/_{\#}\, t \leq r \,{}^{\vee}\!/_{\#}\, (t \# s)$.

($\geq$) Let $s_\beta \# (t \# s) \leq r$. Since $(s_\beta \# t) \# s = s_\beta \# (t \# s)$ we have $(s_\beta \# t) \# s \leq r$, but $r \,{}^{\vee}\!/_{\#}\, s$ is the greatest preradical among $r_\alpha \in \mathbb{PR}$ with the property $r_\alpha \# s \leq r$, so $s_\beta \# t \leq r \,{}^{\vee}\!/_{\#}\, s$, which means that s_β is one of preradicals t_γ. Thus $\vee \{ s_\beta \in \mathbb{PR} \mid s_\beta \# (t \# s) \leq r \} \leq \vee \{ t_\gamma \in \mathbb{PR} \mid t_\gamma \# t \leq r \,{}^{\vee}\!/_{\#}\, s \}$, i.e. $r \,{}^{\vee}\!/_{\#}\, (t \# s) \leq (r \,{}^{\vee}\!/_{\#}\, s) \,{}^{\vee}\!/_{\#}\, t$.

2) The condition $s \geq t$ ensures the existence of the left coquotient $s \,{}^{\vee}\!/_{\#}\, t$. Moreover, since $r \# s \geq s$ we have $r \# s \geq t$, so there exists the left coquotient $(r \# s) \,{}^{\vee}\!/_{\#}\, t$.

By the definition of the left coquotient $s \geq (s \,{}^{\vee}\!/_{\#}\, t) \# t$, from the monotony and the property of associativity of coproduct we obtain $r \# s \geq r \# [(s \,{}^{\vee}\!/_{\#}\, t) \# t] = [r \# (s \,{}^{\vee}\!/_{\#}\, t)] \# t$. Applying Proposition 3.5 we have $(r \# s) \,{}^{\vee}\!/_{\#}\, t \geq r \# (s \,{}^{\vee}\!/_{\#}\, t)$. □

Proposition 3.8. *Let* $r, s, t \in \mathbb{PR}$ *and* $r \geq s$. *The following relations hold:*

1) $(r \,{}^{\vee}\!/_{\#}\, t) \,{}^{\vee}\!/_{\#}\, (s \,{}^{\vee}\!/_{\#}\, t) \geq r \,{}^{\vee}\!/_{\#}\, s$ *or* $(r \,{}^{\vee}\!/_{\#}\, s) \# (s \,{}^{\vee}\!/_{\#}\, t) \leq r \,{}^{\vee}\!/_{\#}\, t$ *for every preradical* $t \leq s$;

2) $(r \# t) \,{}^{\vee}\!/_{\#}\, (s \# t) \geq r \,{}^{\vee}\!/_{\#}\, s$ *for every preradical* $t \in \mathbb{PR}$.

Proof. 1) The condition $r \geq s$ ensures the existence of the left coquotient $r \,{}^{\vee}\!/_{\#}\, s$. In the this case if $t \leq s$, then $r \geq t$, so there exist the left coquotients $s \,{}^{\vee}\!/_{\#}\, t$ and $r \,{}^{\vee}\!/_{\#}\, t$. Moreover, since $r \geq s$, from the monotony of the left coquotient we have $r \,{}^{\vee}\!/_{\#}\, t \geq s \,{}^{\vee}\!/_{\#}\, t$, which ensures the existence of the left coquotien $(r \,{}^{\vee}\!/_{\#}\, t) \,{}^{\vee}\!/_{\#}\, (s \,{}^{\vee}\!/_{\#}\, t)$.

From Proposition 3.5 we have

$$(r \,{}^{\vee}\!/_{\#}\, t) \,{}^{\vee}\!/_{\#}\, (s \,{}^{\vee}\!/_{\#}\, t) \geq r \,{}^{\vee}\!/_{\#}\, s \Leftrightarrow (r \,{}^{\vee}\!/_{\#}\, s) \# (s \,{}^{\vee}\!/_{\#}\, t) \leq r \,{}^{\vee}\!/_{\#}\, t.$$

By the definition of the left coquotient $r \geq (r \,{}^{\vee}\!/_{\#}\, s) \# s$ and $s \geq (s \,{}^{\vee}\!/_{\#}\, t) \# t$. Using the monotony and the property of associativity of coproduct we obtain $r \geq (r \,{}^{\vee}\!/_{\#}\, s) \# s \geq (r \,{}^{\vee}\!/_{\#}\, s) \# [(s \,{}^{\vee}\!/_{\#}\, t) \# t] = [(r \,{}^{\vee}\!/_{\#}\, s) \# (s \,{}^{\vee}\!/_{\#}\, t)] \# t$. Applying Proposition 3.5 we have $r \,{}^{\vee}\!/_{\#}\, t \geq (r \,{}^{\vee}\!/_{\#}\, s) \# (s \,{}^{\vee}\!/_{\#}\, t)$.

2) The condition $r \geq s$ ensures the existence of the left coquotient $r \,{}^{\vee}\!/_{\#}\, s$. Moreover, from the monotony of coproduct we have $r \# t \geq s \# t$ for every preradical $t \in \mathbb{PR}$, which implies the existence of the left coquotient $(r \# t) \,{}^{\vee}\!/_{\#}\, (s \# t)$.

From Proposition 3.5 the relation of this statement is equivalent to the relation $r \# t \geq (r \,{}^{\vee}\!/_{\#}\, s) \# (s \# t)$.

By the definition of the left coquotient $r \geq (r \,{}^{\vee}\!/_{\#}\, s) \# s$, using the monotony and the property of associativity of coproduct of preradicals we obtain $r \# t \geq [(r \,{}^{\vee}\!/_{\#}\, s) \# s] \# t = (r \,{}^{\vee}\!/_{\#}\, s) \# (s \# t)$. □

The following two statements show the relation between the left coquotient with respect to join and the lattice operations of $\mathbb{PR}$.

Proposition 3.9. *(The left distributivity of the left coquotient* $r \,{}^{\vee}\!/_{\#}\, s$ *relative to meet) For any preradical* $s \in \mathbb{PR}$ *and for any family of preradicals* $\{ r_\alpha \in \mathbb{PR} \,|\, r_\alpha \geq s,\ \alpha \in \mathfrak{A} \}$ *the following relation holds:*

$$\Big(\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha \Big) \,{}^{\vee}\!/_{\#}\, s = \bigwedge_{\alpha \in \mathfrak{A}} (r_\alpha \,{}^{\vee}\!/_{\#}\, s).$$

Proof. The relations $r_\alpha \geq s,\ \alpha \in \mathfrak{A}$ ensures the existence of the left coquotients $r_\alpha \,{}^{\vee}\!/_{\#}\, s,\ \alpha \in \mathfrak{A}$. But in this case $\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha \geq s$, so there exists the left coquotient $\Big(\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha \Big) \,{}^{\vee}\!/_{\#}\, s$.

($\leq$) From Lemma 3.1 $\Big(\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha \Big) \,{}^{\vee}\!/_{\#}\, s = \vee \Big\{ t_\beta \in \mathbb{PR} \,|\, t_\beta \# s \leq \bigwedge_{\alpha \in \mathfrak{A}} r_\alpha \Big\}$ and $r_\alpha \,{}^{\vee}\!/_{\#}\, s = \vee \{ r'_\gamma \in \mathbb{PR} \,|\, r'_\gamma \# s \leq r_\alpha \}$.

Let $t_\beta \# s \leq \bigwedge_{\alpha \in \mathfrak{A}} r_\alpha$. Since $\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha \leq r_\alpha$ for every $\alpha \in \mathfrak{A}$ we have $t_\beta \# s \leq r_\alpha$, which means that each preradical t_β is one of preradicals r'_γ. This implies $\vee \Big\{ t_\beta \in \mathbb{PR} \,|\, t_\beta \# s \leq \bigwedge_{\alpha \in \mathfrak{A}} r_\alpha \Big\} \leq \vee \{ r'_\gamma \in \mathbb{PR} \,|\, r'_\gamma \# s \leq r_\alpha \}$ for every $\alpha \in \mathfrak{A}$, therefore $\vee \Big\{ t_\beta \in \mathbb{PR} \,|\, t_\beta \# s \leq \bigwedge_{\alpha \in \mathfrak{A}} r_\alpha \Big\} \leq \bigwedge_{\alpha \in \mathfrak{A}} (\vee \{ r'_\gamma \in \mathbb{PR} \,|\, r'_\gamma \# s \leq r_\alpha \})$, i.e. $\Big(\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha \Big) \,{}^{\vee}\!/_{\#}\, s \leq \bigwedge_{\alpha \in \mathfrak{A}} (r_\alpha \,{}^{\vee}\!/_{\#}\, s)$.

($\geq$) By the definition of the left coquotient we have $r_\alpha \geq (r_\alpha \,{}^{\vee}\!/_{\#}\, s) \# s$ for every $\alpha \in \mathfrak{A}$, which implies that $\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha \geq \bigwedge_{\alpha \in \mathfrak{A}} [(r_\alpha \,{}^{\vee}\!/_{\#}\, s) \# s]$. From the distributivity of coproduct of preradicals relative to meet it follows that $\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha \geq \Big[\bigwedge_{\alpha \in \mathfrak{A}} (r_\alpha \,{}^{\vee}\!/_{\#}\, s) \Big] \# s$. Using Proposition 3.5 we obtain that $\Big(\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha \Big) \,{}^{\vee}\!/_{\#}\, s \geq \bigwedge_{\alpha \in \mathfrak{A}} (r_\alpha \,{}^{\vee}\!/_{\#}\, s)$. □

Proposition 3.10. *In the class* $\mathbb{PR}$ *the following relations are true:*

1) $\Big(\bigvee_{\alpha \in \mathfrak{A}} r_\alpha \Big) \,{}^{\vee}\!/_{\#}\, s \geq \bigvee_{\alpha \in \mathfrak{A}} (r_\alpha \,{}^{\vee}\!/_{\#}\, s)$ *for* $r_\alpha \geq s,\ \alpha \in \mathfrak{A}$*;*

2) $r \,{}^{\vee}\!/_{\#} \Big(\bigwedge_{\alpha \in \mathfrak{A}} s_\alpha \Big) \geq \bigvee_{\alpha \in \mathfrak{A}} (r \,{}^{\vee}\!/_{\#}\, s_\alpha)$ *for* $r \geq s_\alpha,\ \alpha \in \mathfrak{A}$*;*

3) $r \,{}^{\vee}\!/_{\#} \Big(\bigvee_{\alpha \in \mathfrak{A}} s_\alpha \Big) \leq \bigwedge_{\alpha \in \mathfrak{A}} (r \,{}^{\vee}\!/_{\#}\, s_\alpha)$ *for* $r \geq s_\alpha,\ \alpha \in \mathfrak{A}$.

Proof. 1) The conditions $r_\alpha \geq s,\ \alpha \in \mathfrak{A}$ ensure the existence of the left coquotients $r_\alpha \,{}^{\vee}\!/_{\#}\, s,\ \alpha \in \mathfrak{A}$. In this case $\bigvee_{\alpha \in \mathfrak{A}} r_\alpha \geq s$, so there exists the left coquotient $\left(\bigvee\limits_{\alpha\in\mathfrak{A}} r_\alpha\right) {}^{\vee}\!\!/_{\#}\, s$.

By the definition of the left coquotient we have $r_\alpha \geq (r_\alpha \,{}^{\vee}\!\!/_{\#}\, s) \,\#\, s$ for every $\alpha \in \mathfrak{A}$, which implies that $\bigvee\limits_{\alpha\in\mathfrak{A}} r_\alpha \geq \bigvee\limits_{\alpha\in\mathfrak{A}} [(r_\alpha \,{}^{\vee}\!\!/_{\#}\, s) \,\#\, s]$. From the distributivity of coproduct of preradicals relative to join it follows that $\bigvee\limits_{\alpha\in\mathfrak{A}} r_\alpha \geq \left[\bigvee\limits_{\alpha\in\mathfrak{A}} (r_\alpha \,{}^{\vee}\!\!/_{\#}\, s)\right] \#\, s$. Applying Proposition 3.5 we obtain $\left(\bigvee\limits_{\alpha\in\mathfrak{A}} r_\alpha\right) {}^{\vee}\!\!/_{\#}\, s \geq \bigvee\limits_{\alpha\in\mathfrak{A}} (r_\alpha \,{}^{\vee}\!\!/_{\#}\, s)$.

2) The conditions $r \geq s_\alpha,\ \alpha \in \mathfrak{A}$ ensure the existence of the left coquotients $r \,{}^{\vee}\!\!/_{\#}\, s_\alpha,\ \alpha \in \mathfrak{A}$. Moreover, in this case $r \geq \bigwedge\limits_{\alpha\in\mathfrak{A}} s_\alpha$, which implies the existence of the left coquotient $r \,{}^{\vee}\!\!/_{\#} \left(\bigwedge\limits_{\alpha\in\mathfrak{A}} s_\alpha\right)$.

Since $\bigwedge\limits_{\alpha\in\mathfrak{A}} s_\alpha \leq s_\alpha$ for every $\alpha \in \mathfrak{A}$, from Proposition 3.4 we have $r \,{}^{\wedge}\!\!/_{\#} \left(\bigwedge\limits_{\alpha\in\mathfrak{A}} s_\alpha\right) \geq r \,{}^{\wedge}\!\!/_{\#}\, s_\alpha$ for all $\alpha \in \mathfrak{A}$, therefore $r \,{}^{\wedge}\!\!/_{\#} \left(\bigwedge\limits_{\alpha\in\mathfrak{A}} s_\alpha\right) \geq \bigvee\limits_{\alpha\in\mathfrak{A}} (r \,{}^{\wedge}\!\!/_{\#}\, s_\alpha)$.

3) The conditions $r \geq s_\alpha,\ \alpha \in \mathfrak{A}$ ensure the existence of the left coquotients $r \,{}^{\vee}\!\!/_{\#}\, s_\alpha,\ \alpha \in \mathfrak{A}$. In this case $r \geq \bigvee\limits_{\alpha\in\mathfrak{A}} s_\alpha$, which implies the existence of the left coquotient $r \,{}^{\vee}\!\!/_{\#} \left(\bigvee\limits_{\alpha\in\mathfrak{A}} s_\alpha\right)$.

For every $\alpha \in \mathfrak{A}$ we have $\bigvee\limits_{\alpha\in\mathfrak{A}} s_\alpha \geq s_\alpha$. Using the antimonotony in the denominator of the left coquotient it follows that $r \,{}^{\vee}\!\!/_{\#} \left(\bigvee\limits_{\alpha\in\mathfrak{A}} s_\alpha\right) \leq r \,{}^{\vee}\!\!/_{\#}\, s_\alpha$ for all $\alpha \in \mathfrak{A}$, therefore $r \,{}^{\vee}\!\!/_{\#} \left(\bigvee\limits_{\alpha\in\mathfrak{A}} s_\alpha\right) \leq \bigwedge\limits_{\alpha\in\mathfrak{A}} (r \,{}^{\vee}\!\!/_{\#}\, s_\alpha)$. □

In continuation we study some particular cases of the left coquotient with respect to join.

Proposition 3.11. *Let* $r, s \in \mathbb{PR}$. *Then:*

1) $r \,{}^{\vee}\!\!/_{\#}\, r = c\,(r)$ *(see Definition 1.2);*

2) $r \,{}^{\vee}\!\!/_{\#}\, 0 = r$*;*

3) $1 \,{}^{\vee}\!\!/_{\#}\, s = 1$.

Proof. From the definition of the left coquotient we obtain:

1) $r \,{}^{\vee}\!\!/_{\#}\, r = \vee\,\{r_\alpha \in \mathbb{PR} \mid r_\alpha \,\#\, r \leq r\} = \vee\,\{r_\alpha \in \mathbb{PR} \mid r_\alpha \,\#\, r = r\} = c\,(r)$;

2) $r \,{}^{\vee}\!\!/_{\#}\, 0 = \vee\,\{r_\alpha \in \mathbb{PR} \mid r_\alpha \,\#\, 0 \leq r\} = \vee\,\{r_\alpha \in \mathbb{PR} \mid r_\alpha \leq r\} = r$;

3) $1 \,{}^{\vee}\!\!/_{\#}\, s = \vee\,\{r_\alpha \in \mathbb{PR} \mid r_\alpha \,\#\, s \leq 1\} = \vee\,\{r_\alpha \mid r_\alpha \in \mathbb{PR}\} = 1$. □

By Propositions 3.11 we have the following particular cases:

(1) $0 \,{}^{\vee}\!/_{\#}\, 0 = 0$; (2) $1 \,{}^{\vee}\!/_{\#}\, 1 = 1$.

Applying to the relation $r \geq s \geq 0$ Proposition 3.4 we obtain that $r \,{}^{\vee}\!/_{\#}\, r \leq r \,{}^{\vee}\!/_{\#}\, s \leq r \,{}^{\vee}\!/_{\#}\, 0$, i.e. $c\,(r) \leq r \,{}^{\vee}\!/_{\#}\, s \leq r$.

Moreover, from the distributivity of product of preradicals relative to join we obtain $c\,(r) \# r = \left(\bigvee_{r_\alpha \# r = r} r_\alpha \right) \# r = \bigvee_{r_\alpha \# r = r} (r_\alpha \# r) = r$ for every $r \in \mathbb{PR}$.

The following statement shows some properties of the co-equalizer.

Proposition 3.12. *If* $r, s \in \mathbb{PR}$ *and* $r \geq s$, *then:*

1) $c\,(r) \# (r \,{}^{\vee}\!/_{\#}\, s) = r \,{}^{\vee}\!/_{\#}\, s$;
2) $(r \,{}^{\vee}\!/_{\#}\, s) \# c\,(s) = r \,{}^{\vee}\!/_{\#}\, s$;
3) $(r \,{}^{\vee}\!/_{\#}\, s) \,{}^{\vee}\!/_{\#}\, c\,(s) = r \,{}^{\vee}\!/_{\#}\, s$.

Proof. The condition $r \geq s$ ensures the existence of the left coquotient $r \,{}^{\vee}\!/_{\#}\, s$.

1) $c\,(r) \# (r \,{}^{\vee}\!/_{\#}\, s) = (r \,{}^{\vee}\!/_{\#}\, r) \# (r \,{}^{\vee}\!/_{\#}\, s)$. From Proposition 3.8(1) we have $(r \,{}^{\vee}\!/_{\#}\, r) \# (r \,{}^{\vee}\!/_{\#}\, s) \leq r \,{}^{\vee}\!/_{\#}\, s$, but $(r \,{}^{\vee}\!/_{\#}\, r) \# (r \,{}^{\vee}\!/_{\#}\, s) \geq r \,{}^{\vee}\!/_{\#}\, s$, so $c\,(r) \# (r \,{}^{\vee}\!/_{\#}\, s) = r \,{}^{\vee}\!/_{\#}\, s$.

2) $(r \,{}^{\vee}\!/_{\#}\, s) \# c\,(s) = (r \,{}^{\vee}\!/_{\#}\, s) \# (s \,{}^{\vee}\!/_{\#}\, s)$. From Proposition 3.8(1) we have $(r \,{}^{\vee}\!/_{\#}\, s) \# (s \,{}^{\vee}\!/_{\#}\, s) \leq r \,{}^{\vee}\!/_{\#}\, s$, but $(r \,{}^{\vee}\!/_{\#}\, s) \# (s \,{}^{\vee}\!/_{\#}\, s) \geq r \,{}^{\vee}\!/_{\#}\, s$, so $(r \,{}^{\vee}\!/_{\#}\, s) \# c\,(s) = r \,{}^{\vee}\!/_{\#}\, s$.

3) $(r \,{}^{\vee}\!/_{\#}\, s) \,{}^{\vee}\!/_{\#}\, c\,(s) = (r \,{}^{\vee}\!/_{\#}\, s) \,{}^{\vee}\!/_{\#}\, (s \,{}^{\vee}\!/_{\#}\, s)$. Using Proposition 3.8(1) we obtain $(r \,{}^{\vee}\!/_{\#}\, s) \,{}^{\vee}\!/_{\#}\, (s \,{}^{\vee}\!/_{\#}\, s) \geq r \,{}^{\vee}\!/_{\#}\, s$, but from Lemma 3.2 $(r \,{}^{\vee}\!/_{\#}\, s) \,{}^{\vee}\!/_{\#}\, (s \,{}^{\vee}\!/_{\#}\, s) \leq r \,{}^{\vee}\!/_{\#}\, s$, so $(r \,{}^{\vee}\!/_{\#}\, s) \,{}^{\vee}\!/_{\#}\, c\,(s) = r \,{}^{\vee}\!/_{\#}\, s$.

□

We will now consider the case of radical.

Remark 3.13. ([5]) *For every preradical* $r \in \mathbb{PR}$ *we have* $c\,(r)$ *is a radical.*

Proof. $c\,(r) \# c\,(r) = (r \,{}^{\vee}\!/_{\#}\, r) \# (r \,{}^{\vee}\!/_{\#}\, r)$. From Proposition 3.8(1) we have that $(r \,{}^{\vee}\!/_{\#}\, r) \# (r \,{}^{\vee}\!/_{\#}\, r) \leq r \,{}^{\vee}\!/_{\#}\, r$, but $(r \,{}^{\vee}\!/_{\#}\, r) \# (r \,{}^{\vee}\!/_{\#}\, r) \geq r \,{}^{\vee}\!/_{\#}\, r$, hence $c\,(r) \# c\,(r) = c\,(r)$, i.e. $c\,(r)$ is a radical. □

Proposition 3.14. ([5]) *Preradical* r *is a radical if and only if* $c\,(r) = r$.

Proof. ($\Rightarrow$) By Definition 1.2 $c\,(r) = \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \# r = r \}$. Let r is a radical, i.e. $r \# r = r$, hence the preradical r is one of preradicals r_α. Therefore $r \leq \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \# r = r \}$, i.e. $r \leq c\,(r)$, but $c\,(r) \leq r$, so $c\,(r) = r$.

($\Leftarrow$) Let $c\,(r) = r$. Then $r \# r = c\,(r) \# r$, but $c\,(r) \# r = r$, so $r \# r = r$, which means that the preradical r is a radical. □

Moreover, since $c(r) \le r \,{}^{\vee}\!/_{\#}\, s \le r$, if r is a radical, then $r \,{}^{\vee}\!/_{\#}\, s = r$.

Proposition 3.15. *Let $r, s \in \mathbb{PR}$ and s is a radical. Then:*

1) $r \,{}^{\vee}\!/_{\#}\, s \ge s$ *with* $r \ge s$*;*
2) $(r \,{}^{\vee}\!/_{\#}\, s) \,\#\, s = r \,{}^{\vee}\!/_{\#}\, s$ *with* $r \ge s$*;*
3) $(r \,{}^{\vee}\!/_{\#}\, s) \,{}^{\vee}\!/_{\#}\, s = r \,{}^{\vee}\!/_{\#}\, s$ *with* $r \ge s$*;*
4) $(r \,\#\, s) \,{}^{\vee}\!/_{\#}\, s = r \,\#\, s$.

Proof. 1) The condition $r \ge s$ ensures the existence of the left coquotient $r \,{}^{\vee}\!/_{\#}\, s$.

Let $r \ge s$. From the monotony of the left coquotient $r \,{}^{\vee}\!/_{\#}\, s \ge s \,{}^{\vee}\!/_{\#}\, s$. If s is a radical, then $r \,{}^{\vee}\!/_{.}\, s \ge s$.

2) The condition $r \ge s$ ensures the existence of the left coquotient $r \,{}^{\vee}\!/_{\#}\, s$.

If s is a radical, then $(r \,{}^{\vee}\!/_{\#}\, s) \,\#\, s = (r \,{}^{\vee}\!/_{\#}\, s) \,\#\, c(s)$, but Proposition 3.12 $(r \,{}^{\vee}\!/_{\#}\, s) \,\#\, c(s) = r \,{}^{\vee}\!/_{\#}\, s$, so $(r \,{}^{\vee}\!/_{\#}\, s) \,\#\, s = r \,{}^{\vee}\!/_{\#}\, s$.

3) The condition $r \ge s$ ensures the existence of the left coquotient $r \,{}^{\vee}\!/_{\#}\, s$. Moreover, by 1) $r \,{}^{\vee}\!/_{\#}\, s \ge s$, which implies the existence of the left coquotient $(r \,{}^{\vee}\!/_{\#}\, s) \,{}^{\vee}\!/_{\#}\, s$.

From Proposition 3.7(1) $(r \,{}^{\vee}\!/_{\#}\, s) \,{}^{\vee}\!/_{\#}\, s = r \,{}^{\vee}\!/_{\#}\, (s \,\#\, s)$. If s is a radical, then $(r \,{}^{\vee}\!/_{\#}\, s) \,{}^{\vee}\!/_{\#}\, s = r \,{}^{\vee}\!/_{\#}\, s$.

4) From Proposition 3.7(2) $(r \,\#\, s) \,{}^{\vee}\!/_{\#}\, s \ge r \,\#\, (s \,{}^{\vee}\!/_{\#}\, s)$. If s is a radical, then $(r \,\#\, s) \,{}^{\vee}\!/_{\#}\, s \ge r \,\#\, s$. But by Lemma 3.2 $(r \,\#\, s) \,{}^{\vee}\!/_{\#}\, s \le r \,\#\, s$, therefore $(r \,\#\, s) \,{}^{\vee}\!/_{\#}\, s = r \,\#\, s$. □

In the next two statements it is shown when the cancellation properties for the left coquotient hold (see Proposition 3.6).

Proposition 3.16. *Let $r, s \in \mathbb{PR}$. The following conditions are equivalent:*

1) $r = (r \,\#\, s) \,{}^{\vee}\!/_{\#}\, s$*;*
2) $r = t \,{}^{\vee}\!/_{\#}\, s$ *for some preradical* $t \ge s$.

Proof. The condition $t \ge s$ ensures the existence of the left coquotient $t \,{}^{\vee}\!/_{\#}\, s$.

1) $\Rightarrow$ 2) Let $r = (r \,\#\, s) \,{}^{\vee}\!/_{\#}\, s$. Then $r = t \,{}^{\vee}\!/_{\#}\, s$ with $t = r \,\#\, s$.

2) $\Rightarrow$ 1) Let $r = t \,{}^{\vee}\!/_{\#}\, s$ for some preradical $t \ge s$. By the definition of the left coquotient $(t \,{}^{\vee}\!/_{\#}\, s) \,\#\, s \le t$. Applying the monotony of the left coquotient we obtain $[(t \,{}^{\vee}\!/_{\#}\, s) \,\#\, s] \,{}^{\vee}\!/_{\#}\, s \le t \,{}^{\vee}\!/_{\#}\, s$, but from Proposition 3.6 $[(t \,{}^{\vee}\!/_{\#}\, s) \,\#\, s] \,{}^{\vee}\!/_{\#}\, s \ge t \,{}^{\vee}\!/_{\#}\, s$, thus $[(t \,{}^{\vee}\!/_{\#}\, s) \,\#\, s] \,{}^{\vee}\!/_{\#}\, s = t \,{}^{\vee}\!/_{\#}\, s$. Since $t \,{}^{\vee}\!/_{\#}\, s = r$ we have $(r \,\#\, s) \,{}^{\vee}\!/_{\#}\, s = r$. □

Proposition 3.17. *Let $r, s \in \mathbb{PR}$. The following conditions are equivalent:*

1) $r = (r \,{}^{\vee}\!/_{\#}\, s) \,\#\, s$ *with* $r \ge s$*;*
2) $r = t \,\#\, s$ *for some preradical* $t \in \mathbb{PR}$.

Proof. The condition $r \geq s$ ensures the existence of the left coquotient $r \,{}^{\vee}\!/_{\#}\, s$.

1) $\Rightarrow$ 2) Let $r = (r \,{}^{\vee}\!/_{\#}\, s) \# s$. Then $r = t \# s$ with $t = r \,{}^{\vee}\!/_{\#}\, s$.

2) $\Rightarrow$ 1) Let $r = t \# s$ for some preradical $t \in \mathbb{PR}$. By Proposition 3.6 $(t \# s) \,{}^{\vee}\!/_{\#}\, s \geq t$. Using the monotony of coproduct we obtain $[(t \# s) \,{}^{\vee}\!/_{\#}\, s] \# s \geq t \# s$, but from the definition of the left coquotient $[(t \# s) \,{}^{\vee}\!/_{\#}\, s] \# s \leq t \# s$, thus $[(t \# s) \,{}^{\vee}\!/_{\#}\, s] \# s = t \# s$. Since $t \# s = r$, we have $(r \,{}^{\vee}\!/_{\#}\, s) \# s = r$. $\square$

In continuation we indicate the behaviour of the left coquotient $r \,{}^{\vee}\!/_{\#}\, s$ in the case of such types of preradicals as prime, $\wedge$-prime and irreducible.

Proposition 3.18. *If r is a prime preradical, then $r \,{}^{\vee}\!/_{\#}\, s$ is a prime preradical for any preradical $s \leq r$.*

Proof. The condition $r \geq s$ ensures the existence of the left coquotient $r \,{}^{\vee}\!/_{\#}\, s$.

Let the preradical $r \neq 1$ be prime and $t_1 \cdot t_2 \leq r \,{}^{\vee}\!/_{\#}\, s$ for some preradicals $t_1, t_2 \in \mathbb{PR}$. From Proposition 3.5 we have $r \geq (t_1 \cdot t_2) \# s$. By Lemma 1.3(1) $(t_1 \cdot t_2) \# s \geq (t_1 \# s) \cdot (t_2 \# s)$, so $r \geq (t_1 \# s) \cdot (t_2 \# s)$. Since r is prime, it follows that $r \geq t_1 \# s$ or $r \geq t_2 \# s$. Using Proposition 3.5 we obtain $r \,{}^{\vee}\!/_{\#}\, s \geq t_1$ or $r \,{}^{\vee}\!/_{\#}\, s \geq t_2$. So for every $t_1, t_2 \in \mathbb{PR}$ with $t_1 \cdot t_2 \leq r \,{}^{\vee}\!/_{\#}\, s$ we have $t_1 \leq r \,{}^{\vee}\!/_{\#}\, s$ or $t_2 \leq r \,{}^{\vee}\!/_{\#}\, s$, which means that the preradical $r \,{}^{\vee}\!/_{\#}\, s$ is prime. $\square$

Proposition 3.19. *If the preradical r is $\wedge$-prime, then the preradical $r \,{}^{\vee}\!/_{\#}\, s$ is $\wedge$-prime for any preradical $s \leq r$.*

Proof. The condition $r \geq s$ ensures the existence of the left coquotient $r \,{}^{\vee}\!/_{\#}\, s$.

Let r be $\wedge$-prime and $t_1 \wedge t_2 \leq r \,{}^{\vee}\!/_{\#}\, s$ for some preradicals $t_1, t_2 \in \mathbb{PR}$. From Proposition 3.5 we have $r \geq (t_1 \wedge t_2) \# s$. Using the distributivity of coproduct of preradicals relative to meet we obtain $r \geq (t_1 \# s) \wedge (t_2 \# s)$. Since r is $\wedge$-prime, it follows that $r \geq t_1 \# s$ or $r \geq t_2 \# s$. From Proposition 3.5 $r \,{}^{\vee}\!/_{\#}\, s \geq t_1$ or $r \,{}^{\vee}\!/_{\#}\, s \geq t_2$. So for every preradicals $t_1, t_2 \in \mathbb{PR}$ with $t_1 \wedge t_2 \leq r \,{}^{\vee}\!/_{\#}\, s$ we have $t_1 \leq r \,{}^{\vee}\!/_{\#}\, s$ or $t_2 \leq r \,{}^{\vee}\!/_{\#}\, s$, which means that the preradical $r \,{}^{\vee}\!/_{\#}\, s$ is $\wedge$-prime. $\square$

Proposition 3.20. *Let $r, s \in \mathbb{PR}$ and $r = t \# s$ for some preradical $t \in \mathbb{PR}$. If the preradical r is irreducible, then the preradical $r \,{}^{\vee}\!/_{\#}\, s$ is irreducible.*

Proof. By the condition $r = t \# s$ we have $r \geq s$, which ensures the existence of the left coquotient $r \,{}^{\vee}\!/_{\#}\, s$.

Let r be irreducible and $r \,{}^{\vee}\!/_{\#}\, s = t_1 \wedge t_2$ for some preradicals $t_1, t_2 \in \mathbb{PR}$. If $r = t \# s$ for some preradical t, then by Proposition 3.17 $r = (r \,{}^{\vee}\!/_{\#}\, s) \# s$, so $r = (t_1 \wedge t_2) \# s$. Using the distributivity of coproduct of preradicals relative to meet we obtain $r = (t_1 \# s) \wedge (t_2 \# s)$. Since r is irreducible, it follows that $r = t_1 \# s$ or $r = t_2 \# s$. From Proposition 3.5 these relations have the form $r \,{}^{\vee}\!\!/_{\#}\, s \geq t_1$ or $r \,{}^{\vee}\!\!/_{\#}\, s \geq t_2$ respectively. But $r \,{}^{\vee}\!\!/_{\#}\, s = t_1 \wedge t_2$, hence $t_1 \geq r \,{}^{\vee}\!\!/_{\#}\, s$ and $t_2 \geq r \,{}^{\vee}\!\!/_{\#}\, s$. Therefore we obtain $r \,{}^{\vee}\!\!/_{\#}\, s = t_1$ or $r \,{}^{\vee}\!\!/_{\#}\, s = t_2$. So for every preradicals $t_1, t_2 \in \mathbb{PR}$ with $t_1 \wedge t_2 = r \,{}^{\vee}\!\!/_{\#}\, s$ we have $t_1 = r \,{}^{\vee}\!\!/_{\#}\, s$ or $t_2 = r \,{}^{\vee}\!\!/_{\#}\, s$, which means that the preradical $r \,{}^{\vee}\!\!/_{\#}\, s$ is irreducible. □

Moreover, if the preradical r is prime, then its co-equalizer $c\,(r)$ is prime ([6]).

The operation of the left coquotient with respect to join implies some order relations between the associated preradicals.

Corollary 3.21. 1) *For every preradicals* $r, s \in \mathbb{PR}$ *with* $r \geq s$ *the following relations hold:*

$$r \,{}^{\vee}\!\!/_{\#}\, s \leq (r \,{}^{\vee}\!\!/_{\#}\, s) \# s \leq r \leq (r \# s) \,{}^{\vee}\!\!/_{\#}\, s \leq r \# s;$$

2) *If* s *is a radical, then:*

$$s \leq r \,{}^{\vee}\!\!/_{\#}\, s = (r \,{}^{\vee}\!\!/_{\#}\, s) \# s \leq r \leq (r \# s) \,{}^{\vee}\!\!/_{\#}\, s = r \# s$$

for every preradical $r \geq s$. □

In conclusion we can say that in the class $\mathbb{PR}$ of the category R-Mod two new operations are defined and investigated, namely, left quotient with respect to meet and left coquotient with respect to join. These operations are partial in the sense that they do not exist for any two preradicals, but only under certain conditions. They possess a series of properties related with the four operations of the class $\mathbb{PR}$ and are consistent with a series of a notions and constructions from the theory of radicals.

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